Ergodic and non-ergodic clustering of inertial particles

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We compute the fractal dimension of clusters of inertial particles in mixing flows at finite values of Kubo (Ku) and Stokes (St) numbers, by a new series expansion in Ku. At small St, the theory includes clustering by Maxey's non-ergodic 'centrifuge' effect. In the limit of St $\to \infty$ and Ku $\to 0$ (so that Ku^{2}St remains finite) it explains clustering in terms of ergodic 'multiplicative amplification'. In this limit, the theory is consistent with the asymptotic perturbation series in [Duncan *et al.*, Phys. Rev. Lett. **95** (2005) 240602]. The new theory allows to analyse how the two clustering mechanisms compete at finite values of St and Ku. For particles suspended in two-dimensional random Gaussian incompressible flows, the theory yields excellent results for Ku < 0.2 for arbitrary values of St; the ergodic mechanism is found to contribute significantly unless St is very small. For higher values of Ku the new series is likely to require resummation. But numerical simulations show that for Ku $\sim$ St ~ 1 too, ergodic 'multiplicative amplification' makes a substantial contribution to the observed clustering.

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Introduction. The dynamics of independent particles in complex mixing flows is a problem of fundamental importance in the natural sciences, and in technology. The motion of the particles is commonly approximated by

$$\ddot{\boldsymbol{r}} = \gamma[\boldsymbol{u}(\boldsymbol{r},t) - \boldsymbol{v}]\,. \tag{1}$$

Here $\boldsymbol{r}$ is the position of a suspended particle, and $\boldsymbol{v} = \dot{\boldsymbol{r}}$ is its velocity. Dots denote time derivatives, γ is the rate at which the inertial motion is damped relative to the fluid, and $\boldsymbol{u}(\boldsymbol{r},t)$ is the velocity of a randomly mixing or turbulent incompressible flow. It is a surprising fact that even though $\boldsymbol{u}$ is incompressible, the suspended particles may nevertheless cluster together [1]. The effect is illustrated in Fig. 1**a**,**b**. Possible consequences of this phenomenon have been discussed in a wide range of contexts: rain initiation from turbulent clouds [2–4], grain dynamics in circumstellar accretion disks [5, 6], and plankton dynamics [7], to name but a few.

Despite its significance, clustering of particles in mixing flows is still not well understood. Two very different explanations of the phenomenon have been put forward. Maxey [1] discussed the problem in the limit of small inertia, corresponding to small values of the 'Stokes number' St $= (\gamma\tau)^{-1}$. Here τ is the relevant characteristic time scale of the flow (the Kolmogorov time in turbulent flows, for example). For $0 <$ St $\ll 1$, the particles are argued to be 'centrifuged' out of regions of high vorticity of $\boldsymbol{u}(\boldsymbol{r},t)$. The approach rests on instantaneous correlations between particle positions and fluid velocities. It has been refined by many authors [8–10] and is commonly used to interpret results of experiments [11, 12], and of direct numerical simulations [13]. But the 'centrifuge' mechanism relies on a small-St expansion, while clustering in turbulent flows is observed to be strongest and thus of most interest at St ~ 1.

A very different clustering mechanism was proposed [14] in the limit of large St and small 'Kubo numbers'.

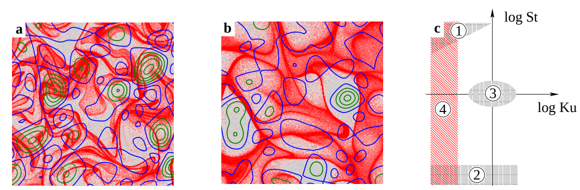

FIG. 1: **a** Clustering of particles in a two-dimensional incompressible flow $\boldsymbol{u}(\boldsymbol{r},t) = \boldsymbol{\nabla} \wedge \psi(\boldsymbol{r},t)\mathbf{e}_3$. Here $\mathbf{e}_3$ is the unit vector $\perp$ to the plane. The Gaussian random function $\psi(\boldsymbol{r},t)$ satisfies $\langle\psi\rangle = 0$ and $\langle\psi(\boldsymbol{r},t)\psi(\mathbf{0},0)\rangle = (u_0^2\eta^2/2)\,\exp\left[-|\boldsymbol{r}|^2/(2\eta^2) - |t|/\tau\right]$. Green contours correspond to high vorticity of $\boldsymbol{u}(\boldsymbol{r},t)$, blue contours to high strains. Particle number density: white (low density) to red (high density). Parameters: Ku $= 0.1$,St $= 10$,$t = 165\tau$. **b**, same but for Ku $= 10$, St $= 0.025$, $t = 0.32\tau$. **c** Parameter plane for inertial particles in mixing flows. Region 1: clustering is caused by ergodic 'multiplicative amplification', see text. Region 2: the non-ergodic 'centrifuge' mechanism is important. Turbulent flows correspond to Ku ~ 1, strong clustering is observed for St ~ 1, region 3. In region 4 the new perturbation expansion is accurate (schematic).

The Kubo number [14, 15] Ku $= u_0\tau/\eta$ characterises fluctuations of $\boldsymbol{u}(\boldsymbol{r},t)$ (u_0 and η being its characteristic velocity and length scales). In the limit St $\to \infty$ (and Ku $\to 0$ so that $\epsilon^2 \equiv$ Ku^{2}St$/2$ remains constant), the particles experience the velocity field as a white-noise signal, and sample it in an ergodic fashion: the fluctuations of $\boldsymbol{u}(\boldsymbol{r}_t,t)$ (and its derivatives) along a particle trajectory $\boldsymbol{r}_t$ are indistinguishable from the fluctuations of $\boldsymbol{u}(\boldsymbol{r}_0,t)$ at the fixed position $\boldsymbol{r}_0$. This case corresponds to region 1 in the phase diagram Fig. 1**c**, and in this limit the instantaneous configuration of $\boldsymbol{u}(\boldsymbol{r}_t,t)$ is irrelevant to the dynamics of the suspended particles. But they may nevertheless cluster by the mechanism of 'multiplicative amplification': small line-, area-, and volume elements randomly expand and contract. Depending upon whether the random product of expansion and contraction fac-

tors increases or decreases as $t \to \infty$, one may observe fractal clustering in region 1. The fractal dimension $d_{\rm L}$ is determined by the history of these factors. It can be computed in terms of 'Lyapunov exponents' [14, 16].

Figs. 1**a**,**b** show both mechanisms at work: at large values of St, (region 1 in Fig. 1**c**) there is no discernible influence of the instantaneous $\boldsymbol{u}(\boldsymbol{r}_t,t)$ on the particle distribution. At small St, (region 2 in Fig. 1**c**), by contrast, the particles are seen to avoid regions of high vorticity (Fig. 1**b**, similar to Fig. 8 in [17]). In short, in limiting cases (regions 1 and 2 of Fig. 1**c**) the mechanisms of clustering are understood. But how ergodic and non-ergodic effects compete in the major part of the phase diagram Fig. 1**c** is not known (in particular not for the experimentally most relevant region 3 where Ku and St are of order unity). In [18] non-ergodic effects were characterised by correlating the degree of clustering with the probability of particles avoiding rotational regions of the flow. The interpretation of these numerical results, however, is complicated by the fact that this probability is significantly enhanced even when clustering is weak. In order to understand the importance of non-ergodic and ergodic effects, an analytical theory is required, valid at finite Stokes and Kubo numbers.

Summary. Here we derive a perturbation expansion for the Lyapunov exponents of particles in random flows, valid at finite St and Ku. We compute the Lyapunov fractal dimension $d_{\rm L}$, and characterise fractal clustering in terms of the 'dimension deficit' $\Delta_{\rm L} = d - d_{\rm L}$. For particles suspended in two-dimensional random Gaussian incompressible flows, the new theory yields reliable results for $\mathrm{Ku} < 0.2$ for arbitrary values of St, region 4 in Fig. 1**c** (schematic). We find, first, that for small values of St, the 'centrifuge' mechanism dominates, and $\Delta_{\rm L} = 6\mathrm{Ku}^2\mathrm{St}^2$, consistent with [20, 21]. Second, in the limit of $\mathrm{Ku} \to 0$ at finite values of St, non-ergodic effects remain important. Third, in region 1 of Fig. 1**c**, clustering is found to be entirely due to ergodic 'multiplicative amplification' [14], and $\Delta_{\rm L} = 12\epsilon^2 \propto \mathrm{St}$. Fourth, in general we find that the ergodic mechanism contributes substantially to clustering, unless St is very small. Fifth we show by numerical simulations of the model that at $\mathrm{Ku} \sim 1$ and $\mathrm{St} \sim 1$, ergodic 'multiplicative amplification' makes a substantial contribution to the observed clustering.

Method and results. Eq. (1) cannot be explicitly solved, since $\boldsymbol{u}$ depends upon the particle position at time t. The implicit solution of Eq. (1) becomes (dimensionless variables $\boldsymbol{r}' = \boldsymbol{r}/\eta$, $t' = t/\tau$, $\boldsymbol{v}' = \boldsymbol{v}/u_0$, and $\boldsymbol{u}' = \boldsymbol{u}/u_0$)

$$
\begin{aligned}
\delta\boldsymbol{r}_t \;\equiv\; \boldsymbol{r}_t - \boldsymbol{r}_0 = \mathrm{Ku}\Big[&\mathrm{St}(1-\mathrm{e}^{-t/\mathrm{St}})\boldsymbol{v}_0 \\
&+\mathrm{St}^{-1}\int_0^t \mathrm{d}t_1 \int_0^{t_1}\mathrm{d}t_2\, \mathrm{e}^{-(t_1-t_2)/\mathrm{St}}\boldsymbol{u}(\boldsymbol{r}_{t_2},t_2)\Big]\,.
\end{aligned}
\tag{2}
$$

Here and in the following the primes are omitted. We seek an approximate solution by expanding $\boldsymbol{u}(\boldsymbol{r}_t,t)$ in powers of $\delta\boldsymbol{r}_t$. Since according to Eq. (2), $\delta\boldsymbol{r}_t$ is of order Ku, iteration generates an expansion of $\boldsymbol{u}(\boldsymbol{r}_t,t)$ in powers of Ku. To second order, for example, we find

$$
\begin{aligned}
u_\alpha(\boldsymbol{r}_t,t) &= u_\alpha(\boldsymbol{r}_0,t) + \frac{\mathrm{Ku}}{\mathrm{St}}\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2 e^{-(t_1-t_2)/\mathrm{St}}\sum_\beta \frac{\partial u_\alpha}{\partial r_\beta}(\boldsymbol{r}_0,t)u_\beta(\boldsymbol{r}_0,t_2)\\
&+\frac{\mathrm{Ku}^2}{\mathrm{St}^2}\int_0^t\mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\int_0^{t_2}\mathrm{d}t_3\int_0^{t_3}\mathrm{d}t_4 e^{-(t_1-t_2+t_3-t_4)/\mathrm{St}}\sum_{\beta,\delta}\frac{\partial u_\alpha}{\partial r_\beta}(\boldsymbol{r}_0,t)\frac{\partial u_\beta}{\partial r_\delta}(\boldsymbol{r}_0,t_2)u_\delta(\boldsymbol{r}_0,t_4)\\
&+\frac{1}{2}\frac{\mathrm{Ku}^2}{\mathrm{St}^2}\int_0^t\mathrm{d}t_1\int_0^{t}\mathrm{d}t_2\int_0^{t_1}\mathrm{d}t_3\int_0^{t_2}\mathrm{d}t_4 e^{-(t_1+t_2-t_3-t_4)/\mathrm{St}}\sum_{\beta,\delta}\frac{\partial^2 u_\alpha}{\partial r_\beta\partial r_\delta}(\boldsymbol{r}_0,t)u_\beta(\boldsymbol{r}_0,t_3)u_\delta(\boldsymbol{r}_0,t_4) + O(\mathrm{Ku}^3)\,.
\end{aligned}
\tag{3}
$$

Here Greek indices denote the components of $\boldsymbol{u}$, and for our purposes $\boldsymbol{v}_0$ can be set to zero. The coefficients in (3) are expressed in terms of $\boldsymbol{u}$ and its derivatives at the fixed position $\boldsymbol{r}_0$, with known statistical properties. This procedure can in principle be extended to any order in Ku, but in practice it is limited by the number of nested integrals appearing in (3) for higher orders. A C-program was written to symbolically evaluate the integrals. One may expand other functionals of the particle trajectories, such as the strain matrix $\mathbb{A}(\boldsymbol{r}_t,t)$ with elements $A_{\alpha\beta} = \partial u_\alpha/\partial r_\beta$. Previous analytical results on the clustering of inertial particles [14, 19] rest on the 'ergodic assumption' that the distribution of the strain matrix $\mathbb{A}(\boldsymbol{r}_t,t)$ at the particle position $\boldsymbol{r}_t$ can be approximated by its distribution at $\boldsymbol{r}_0$. This is satisfied in region 1, but what are the corrections outside this region? For the two-dimensional incompressible ($\mathrm{Tr}\mathbb{A} = 0$) random flow described in Fig. 1 we find:

$$
\begin{aligned}
\overline{\mathrm{Tr}\mathbb{A}^2} \;\equiv\; \Big\langle \lim_{T\to\infty}\frac{1}{T}\int_0^T\mathrm{d}t\,\mathrm{Tr}\mathbb{A}^2(\boldsymbol{r}_t,t)\Big\rangle &= \frac{6\mathrm{Ku}^2\mathrm{St}}{(1+\mathrm{St})^2(1+2\mathrm{St})}\\
&-\frac{2\mathrm{Ku}^4\mathrm{St}(4+52\mathrm{St}+293\mathrm{St}^2+548\mathrm{St}^3+297\mathrm{St}^4)}{(1+\mathrm{St})^4(2+\mathrm{St})(1+2\mathrm{St})^2(1+3\mathrm{St})}
\end{aligned}
\tag{4}
$$

to order Ku^4. The average in (4) consists of a long-time average along the particle trajectory $\boldsymbol{r}_t$, and an average over initial conditions $\boldsymbol{r}_0$ (denoted by $\langle\cdots\rangle$). At finite values of the Kubo number, $\overline{\mathrm{Tr}\mathbb{A}^2}$ differs from its ergodic average (which vanishes in incompressible flows): the dynamics is not strictly ergodic. In compressible flows, we find $\overline{\mathrm{Tr}\mathbb{A}}\neq 0$ (the ergodic average still vanishes). This is consistent with a result [22] for the average strain in the advective limit of a one-dimensional (compressible) model. It was shown in [22] that the average strain must be taken into account to obtain the known result [9] for the advective Lyapunov exponent in this model.

The question is now: how does non-ergodicity affect the spatial distribution of the particles? The latter is characterised by the Lyapunov exponents of the particle flow, obtained by linearising Eq. (1). The maximal exponent λ_1 is given by

$$\lambda_1 \;=\; \mathrm{Ku}\lim_{T\to\infty}\frac{1}{T}\int_0^T\!\mathrm{d}t\,\boldsymbol{n}_1(\boldsymbol{r}_t,t)\cdot\mathbb{Z}(\boldsymbol{r}_t,t)\boldsymbol{n}_1(\boldsymbol{r}_t,t)\,,\quad(5)$$
$$\dot{\mathbb{Z}} \;=\; \mathrm{St}^{-1}(\mathbb{A}-\mathbb{Z})-\mathrm{Ku}\,\mathbb{Z}^2,\;\dot{\boldsymbol{n}}_1=\mathrm{Ku}\big(\boldsymbol{n}_2\cdot\mathbb{Z}\boldsymbol{n}_1\big)\boldsymbol{n}_2\,.\quad(6)$$

Here $\boldsymbol{n}_1$ ($\boldsymbol{n}_2$) is the unit vector in the $\delta\boldsymbol{r}$ ($\delta\dot{\boldsymbol{r}}$)-direction, and $\mathbb{Z}$ is the matrix with elements $Z_{\alpha\beta}=\partial v_\alpha/\partial r_\beta$. In region 1, Eqs. (5,6) were solved in [14, 19]. At finite values of Ku, we compute λ_1 by generalising the procedure that led to Eq. (3). Starting from the implicit solution (2) of (1), and the implicit solutions of (6):

$$\begin{aligned}\mathbb{Z}(\boldsymbol{r}_t,t) &= \mathrm{e}^{-t/\mathrm{St}}\mathbb{Z}(\boldsymbol{r}_0,0)+\int_0^t\!\mathrm{d}t_1\mathrm{e}^{-\frac{t-t_1}{\mathrm{St}}}\big[\mathbb{A}(\boldsymbol{r}_{t_1},t_1)/\mathrm{St}\\&\qquad-\mathrm{Ku}\,\mathbb{Z}(\boldsymbol{r}_{t_1},t_1)^2\big]\,,\qquad(7)\\ \boldsymbol{n}_1(\boldsymbol{r}_t,t) &= \boldsymbol{n}_1(\boldsymbol{r}_0,0)+\mathrm{Ku}\int_0^t\!\mathrm{d}t_1[\boldsymbol{n}_2(\boldsymbol{r}_{t_1},t_1)\cdot\mathbb{Z}(\boldsymbol{r}_{t_1},t_1)\\&\qquad\times\boldsymbol{n}_1(\boldsymbol{r}_{t_1},t_1)]\boldsymbol{n}_2(\boldsymbol{r}_{t_1},t_1)\,,\qquad(8)\end{aligned}$$

we expand $\mathbb{A}(\boldsymbol{r}_t,t)$, $\mathbb{Z}(\boldsymbol{r}_t,t)$, $\boldsymbol{n}_1(\boldsymbol{r}_t,t)$, and $\boldsymbol{n}_2(\boldsymbol{r}_t,t)$ in powers of $\delta\boldsymbol{r}_t$. Iterating and averaging along particle trajectories as well as over initial conditions yields an expansion of λ_1 in powers of Ku, with St-dependent coefficients. The sum $\lambda_1+\lambda_2=\mathrm{Ku}\,\overline{\mathrm{Tr}\mathbb{Z}}$ is computed in a similar fashion. For particles in a two-dimensional incompressible random Gaussian flow (c.f. Fig. 1) we find

$$\begin{aligned}\lambda_1 &= \mathrm{Ku}^2-\mathrm{Ku}^4\frac{6+16\mathrm{St}+16\mathrm{St}^2+15\mathrm{St}^3+5\mathrm{St}^4}{(1+\mathrm{St})^3}+\mathrm{Ku}^6\Big[\frac{1692+16464\mathrm{St}+68987\mathrm{St}^2+165269\mathrm{St}^3+258832\mathrm{St}^4}{6(1+\mathrm{St})^5(2+\mathrm{St})^2(1+2\mathrm{St})^2}\\&\quad+\frac{301534\mathrm{St}^5+296820\mathrm{St}^6+247404\mathrm{St}^7+153480\mathrm{St}^8+62136\mathrm{St}^9+14400\mathrm{St}^{10}+1440\mathrm{St}^{11}}{6(1+\mathrm{St})^5(2+\mathrm{St})^2(1+2\mathrm{St})^2}\Big]\,,\qquad(9)\\ \lambda_1+\lambda_2 &= -6\mathrm{Ku}^4\frac{\mathrm{St}^2(1+3\mathrm{St}+\mathrm{St}^2)}{(1+\mathrm{St})^3}\qquad(10)\\&\quad+2\mathrm{Ku}^6\mathrm{St}^2\frac{8+92\mathrm{St}+598\mathrm{St}^2+2509\mathrm{St}^3+5760\mathrm{St}^4+7176\mathrm{St}^5+5052\mathrm{St}^6+2076\mathrm{St}^7+480\mathrm{St}^8+48\mathrm{St}^9}{(1+\mathrm{St})^5(2+\mathrm{St})^2(1+2\mathrm{St})^2}\,,\end{aligned}$$

to order Ku^6. This is our main result. Eq. (9) yields the ergodic expansion [19] $\lambda_1/(\gamma\tau)=2\epsilon^2-20\epsilon^4+480\epsilon^6+\ldots$ in region 1.

As $\mathrm{St}\to 0$, Eq. (10) reflects Maxey's non-ergodic centrifuge mechanism: According to (1), a particle is advected by an effective velocity field $\boldsymbol{v}$ with compressibility $\nabla\cdot\boldsymbol{v}=\mathrm{Tr}\mathbb{Z}$. Maxey's result [1, 2] is obtained by expanding $\mathbb{Z}\approx\mathbb{Z}^{(0)}+Z^{(1)}\mathrm{St}$ in Eq. (6). One finds: $\nabla\cdot\boldsymbol{v}=-\mathrm{Ku}\,\mathrm{St}\,\mathrm{Tr}\mathbb{A}^2\big|_{\mathrm{St}=0}$ (note that $\mathbb{A}(\boldsymbol{r}_t,t)$ depends upon the Stokes number because the particle trajectory $\boldsymbol{r}_t$ depends upon St). This result shows that particles tend to aggregate in regions of high strain or low vorticity. However, since the velocity field is homogeneous, this lowest-order term averages to zero in incompressible flows. Expanding $\mathbb{Z}$ to second order in St, one finds $\overline{\nabla\cdot\boldsymbol{v}}=-\mathrm{Ku}\,\mathrm{St}^2\,\partial_{\mathrm{St}}\overline{\mathrm{Tr}(\mathbb{A}^2)}\Big|_{\mathrm{St}=0}$. Inserting Eq. (4) yields $\lambda_1+\lambda_2=\mathrm{Ku}\,\overline{\boldsymbol{\nabla}\cdot\boldsymbol{v}}=-6\mathrm{Ku}^4\mathrm{St}^2+4\mathrm{Ku}^6\mathrm{St}^2+O(\mathrm{Ku}^8)$, consistent with the $\mathrm{St}\to 0$ limit of (10).

We conclude that our new expansion (9,10) of the Lyapunov exponents correctly describes the different clustering mechanisms in the advective and ergodic regions (Fig. 1**c**): the 'centrifuge' effect and ergodic 'multiplicative amplification'. More importantly, Eqs. (9,10) allow to determine how the relative importance of the two mechanisms depends on Ku and St, as follows.

In two-dimensional incompressible flows, the fractal dimension deficit is given by $\Delta_{\mathrm{L}}=(\lambda_1+\lambda_2)/\lambda_2$. Fig. 2**a** shows that the new theory explains the limiting cases in region 1 in the parameter plane ($\Delta_{\mathrm{L}}=6\,\mathrm{Ku}^2\,\mathrm{St}=12\,\epsilon^2$), and in region 2 ($\Delta_{\mathrm{L}}=6\,\mathrm{Ku}^2\mathrm{St}^2$). The theory also explains the cross-over between these two behaviours and compares well with results of numerical simulations. In order to further increase the accuracy, more terms than

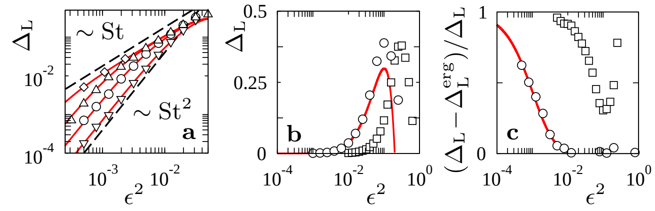

FIG. 2: **a** Fractal dimension deficit as a function of $\epsilon^2 = \mathrm{Ku}^2\mathrm{St}/2$. Numerical simulations of the model described in Fig. 1 (Ku = 0.02 (◊), Ku = 0.05 (△), Ku = 0.1 (○), and Ku = 0.2 (▽); theory according to Eqs. (9,10), solid lines, and limiting behaviours $\Delta_L \propto \mathrm{St}$ and $\Delta_L \propto \mathrm{St}^2$ (dashed lines). **b** Same but for larger values of $\epsilon^2 = \mathrm{Ku}^2\mathrm{St}/2$. Numerics, Ku = 0.1 (○) and Ku = 1 (□); theory, Eqs. (9,10), Ku = 0.1 (solid line). **c** Relative importance of non-ergodic and ergodic contributions (see text), symbols and parameters as in **b**.

computed in Eqs. (9,10) must be included. The series is likely to be asymptotic requiring re-summation, and there may be additional non-analytic contributions [14, 19]. We note that for a slightly different estimate of the fractal dimension (the 'correlation dimension'), the lowest-order behaviour of the dimension deficit in region 2, $\Delta_C \propto \mathrm{St}^2$, was computed using different methods in [9, 10, 23–25]. For the model described in Fig. 1, we find $\Delta_C = 12\,\mathrm{Ku}^2\mathrm{St}^2$ to lowest order in Ku and in St in region 2, and $\Delta_C = 24\,\epsilon^2$ in region 1 (see also [26]).

These results raise the question: how important are non-ergodic contributions to fractal clustering at larger values of St and at finite Kubo numbers? The answer is summarised in Fig. 2**b**,**c** showing the dimension deficit Δ_L compared to an ergodic approximation, Δ_L^{erg}. The latter incorporates finite-time correlations of the velocity field $\boldsymbol{u}$ but neglects non-ergodic effects. Ergodic approximations to the Lyapunov exponents and to Δ_L (referred to as $\lambda_{1,2}^{erg}$ and Δ_L^{erg}) are obtained by expanding Eqs. (5,6) as before, but replacing $\mathbb{A}(\boldsymbol{r}_t, t)$ in (6) with $\mathbb{A}(\boldsymbol{r}_0, t)$. The resulting analytical expressions for $\lambda_{1,2}^{erg}$ are determined by the fluctuations of $\mathbb{A}(\boldsymbol{r}_0, t)$. This is in contrast to $\lambda_{1,2}$, Eqs. (9,10), which are determined by the fluctuations of $\boldsymbol{u}(\boldsymbol{r}_0, t)$ and its derivatives. The ergodic approximation allows for finite values of Ku and St but must fail in the limit $\mathrm{St} \to 0$, since the 'centrifuge' mechanism is not accounted for. In particular, to lowest order in Ku the exponents λ_1^{erg} and λ_2^{erg} are found to depend upon St. The exact exponents (9,10) by contrast, are independent of St to lowest order in Ku: $\lambda_1 = \mathrm{Ku}^2$ and $\lambda_2 = -\mathrm{Ku}^2$. This implies in particular that earlier results for the Lyapunov exponents in region 1 [14, 19] are in fact exact to lowest order in Ku for arbitrary values of St. This is due to the cancellation of two errors: neglecting non-ergodic effects, and neglecting finite-time correlations.

Non-ergodic effects dominate the clustering when $(\Delta_L - \Delta_L^{erg})/\Delta_L$ is close to unity (they are negligible when this ratio is close to zero). We have determined $\lambda_{1,2}^{erg}$ and Δ_L^{erg} to order Ku^6 [27]. We have also performed computer simulations of this 'ergodic model' by calculating monodromy matrices (Eq. (30) in [21]) with $\mathbb{A}(\boldsymbol{r}_0, t)$ evaluated at the fixed position ver_0. Fig. 2**c** shows that for Ku = 0.1, non-ergodic effects dominate at small values of ϵ^2 but are negligible when clustering is largest, near the peak in Δ_L shown in Fig. 2**b**. Also shown are the analytical result for Δ_L derived from Eqs. (9,10), and the corresponding expansion of $(\Delta_L - \Delta_L^{erg})/\Delta_L$. We observe good agreement. For Ku = 1, by contrast, the first terms in the perturbation expansion (9,10) do not give reliable results. But computer simulations show that non-ergodic effects are present for the whole range of Stokes numbers displayed in Fig. 2**c**. However, Fig. 2**c** also clearly shows that both mechanisms contribute in region 3. We emphasise that ergodic clustering by 'multiplicative amplification' makes a substantial contribution in this region, $(\Delta_L - \Delta_L^{erg})/\Delta_L \approx 0.3$.

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